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Two-Phase MFD Flows in Orthogonal Curvilinear Coordinate System

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Steady flow of an inviscid, incompressible two-phase magnetofluid with infinite electrical conductivity is treated. With one ignorable coordinate in a general orthogonal curvilinear system, general solutions of the equations, considering number density N constant throughout the motion, are obtained.

1. INTRODUCTION

Multiphase fluid phenomena are of extreme importance in various field of science and technology, such as geophysics, nuclear engineering, chemical engineering, etc. In recent years, a considerable amount of work has been devoted to dusty fluid flows due to the importance of such studies in many physical applications, ranging from fluidization problems to high-speed and dust hypersonic flows. Multiphase fluid systems are concerned with the motion of a liquid or gas containing immiscible inert particles. Of multiphase fluid systems observed in nature, blood flow, flow in a rocket chamber, dust in gas cooling systems to enhance the heat transfer process, movement of inert particles in the atmosphere, and movement of sand and other suspended particles in beaches are the most common examples. Naturally, studies of these systems are mathematically interesting and physically useful. The presence of particles in a homogeneous fluid makes the dynamical study of a flow problem quite complicated. However, these problems are usually investigated under various simplifying assumptions.

Saffman (1962) has formulated the equations of motion of a dusty fluid represented in terms of a large number density $N(x, t)$ of very small spherical inert particles whose volume concentration is small enough to be neglected. It is assumed that the density of the dust particles is large

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compared with the fluid density, so that the mass concentration of the particles is an appreciable fraction of unity. In this formulation, Saffman also assumed that the individual particles of dust are so small that Stokes' law of resistance between the particles and the fluid remains valid. Using the model of Saffman, several authors, including Michael and Miller (1966), Liu (1967), Debnath and Basu (1975), Barron and Hamdan (1989), and Thakur and Mishra (1988, 1989), have investigated various aspects of hydrodynamics and hydromagnetic two-phase fluid flows.

In this paper, considering number density N constant throughout the motion, we obtain the general solutions of steady, inviscid, and incompressible hydromagnetic two-phase flows with one ignorable coordinate in an orthogonal curvilinear coordinate system.

2. FUNDAMENTAL EQUATIONS

We take the following assumptions to describe the motion of the mixed system of fluid and dust particles:

(a) The fluid is incompressible and inviscid apart from fluid and dust particle interaction.
(b) The particles are spheres and uniform in size. The number of particles is so large that the system of particles can be considered as a continuous medium. The velocity fluctuation of the dust particle at a given point, the interaction between the dust particles, and the bulk concentration of the dust particle are negligible.
(c) The fluid and dust particle interaction follows Stokes' drag law.

Under these assumptions, the equations describing steady motion of the system of the fluid and dust particles with infinite electrical conductivity in the presence of a magnetic field are

$$\operatorname{div} \mathbf{u} = 0 \tag{2.1}$$

$$(\mathbf{u} \cdot \operatorname{grad})\mathbf{u} = \frac{1}{\rho} \nabla p + \frac{KN}{\rho}(\mathbf{v} - \mathbf{u}) + \frac{\mu}{\rho} \operatorname{curl} \mathbf{H} \times \mathbf{H} \tag{2.2}$$

$$\operatorname{curl}(\mathbf{u} \times \mathbf{H}) = \mathbf{0} \tag{2.3}$$

$$\operatorname{div}(N\mathbf{v}) = 0 \tag{2.4}$$

$$m(\mathbf{v} \cdot \operatorname{grad})\, \mathbf{v} = K(\mathbf{u} - \mathbf{v}) \tag{2.5}$$

$$\operatorname{div} \mathbf{H} = 0 \tag{2.6}$$

where $\mathbf{u}$, $\mathbf{v}$, $\mathbf{H}$, ρ, and N are the fluid-phase velocity, the dust-phase velocity, the magnetic field, the fluid density, and the number density of dust particles, respectively; P is the fluid pressure, μ is the coefficient of magnetic permeability, K is the Stokes coefficient of resistance, and m is the mass of a single dust particle.

In this paper we take N to be constant throughout the motion. Now we consider that the fluid-phase velocity $\mathbf{u}$ and dust-phase velocity $\mathbf{v}$ are everywhere parallel, and hence we have

$$\mathbf{v}\times\mathbf{u}=0 \tag{2.7}$$

3. A JACOBIAN FORMULATION OF THE EQUATIONS

We consider a general orthogonal curvilinear coordinate system $\hat{e}_1$, $\hat{e}_2$, $\hat{e}_3$ with line elements $h_1\,(x_1, x_2)$, $h_2(x_1, x_2)$, $h_3(x_1, x_2)$, where coordinate x_3 is ignorable. The set of coordinates (x_1, x_2, x_3) can be any one of the sets of the familiar Cartesian (x, y, z), cylindrical (z, ω, ϕ), spherical (r, θ, ϕ), spheroidal (ξ, η, ϕ), toroidal (u, v, ϕ), bipolar (u, ν, z), etc, coordinates. The particular geometry of the physical problem under consideration will indicate the choice of a suitable system. In toroidal coordinates,

$$x_1=u, \qquad x_2=v, \qquad x_3=\phi, \qquad x=\frac{a\sinh v\cos\phi}{\cosh v-\cos u}$$

$$y=\frac{\sinh v\sin\phi}{\cosh v-\cos u}, \qquad z=\frac{\sin u}{\cosh v-\cos u}$$

with the line elements

$$h_1^2(u, v)=h_2^2(u, v)=\frac{a^2}{(\cosh v-\cos u)^2}$$

$$h_3^2(u, v)=\frac{a^2\sinh^2 v}{(\cosh v-\cos u)^2}$$

or, in oblate spheroidal coordinates, $x_1=\zeta$, $x_2=\eta$, $x_3=\phi$, $x=a\cosh\xi\cos\eta\cos\phi$, $y=a\cosh\xi\cos\eta\sin\phi$, and $z=a\sinh\xi\sin\eta$, with the line elements

$$h_1^2(\xi, \eta)=h_1^2(\xi, \eta)=a^2(\sinh^2\xi+\sin^2\eta)$$

$$h_3^2(\xi, \eta)=a^2\cosh^2\xi\cos^2\eta$$

are examples of orthogonal curvilinear coordinates x_1, x_2, x_3 where one of them $(x_3=\phi)$ is ignorable. Hence the vectors $\mathbf{u}$ and $\mathbf{v}$ can be written as

$$\mathbf{u}=\hat{e}_1u_1+\hat{e}_2u_2+\hat{e}_3u_3 \tag{3.1}$$

$$\mathbf{v}=\hat{e}_1v_1+\hat{e}_2v_2+\hat{e}_3v_3 \tag{3.2}$$

where u_1, u_2, u_3 and v_1, v_2, v_3 are the curvilinear components of $\mathbf{u}$ and $\mathbf{v}$ along the unit vectors $\hat{e}_1$, $\hat{e}_2$, $\hat{e}_3$, respectively.

The two conservation laws, equation (2.1) for the fluid phase and equation (2.4) for the dust phase, have the following forms:

$$\frac{\partial}{\partial x_1}(h_2 h_3 u_1) + \frac{\partial}{\partial x_2}(h_1 h_3 u_2) = 0 \tag{3.3}$$

$$\frac{\partial}{\partial x_1}(h_2 h_3 v_1) + \frac{\partial}{\partial x_2}(h_1 h_3 v_2) = 0 \tag{3.4}$$

Equations (3.3) and (3.4) imply, respectively, the existence of functions $\psi(x_1, x_2)$ and $\phi(x_1, x_2)$ such that (Tsinganos, 1982)

$$h_2 h_3 u_1 = \frac{\partial \psi(x_1, x_2)}{\partial x_1}, \qquad h_1 h_3 u_2 = \frac{-\partial \psi(x_1, x_2)}{\partial x_2} \tag{3.5}$$

$$h_2 h_3 v_1 = \frac{\partial \phi(x_1, x_2)}{\partial x_1}, \qquad h_1 h_3 v_2 = -\frac{\partial \phi(x_1, x_2)}{\partial x_2} \tag{3.6}$$

We notice that the contour $\psi(x_1, x_2) = \text{const}$ or $\phi(x_1, x_2) = \text{const}$ (which identify the projections of the fields on the surface $x_3 = \text{const}$) define surfaces made up of fluid-phase streamlines or dust-phase streamlines. We will call the surfaces $\psi(x_1, x_2) = \text{const}$ the fluid-phase stream surfaces and the surface $\phi(x_1, x_2) = \text{const}$ the dust-phase stream surfaces.

Using the results (3.5) and (3.6), we can write $\mathbf{u}$ and $\mathbf{v}$ from (3.1) and (3.2) as

$$\mathbf{u}(x_1, x_2) = \frac{\hat{e}_1}{h_2 h_3} \frac{\partial \psi(x_1, x_2)}{\partial x_2} - \frac{\hat{e}_2}{h_1 h_3} \frac{\partial \psi(x_1, x_2)}{\partial x_1} + \hat{e}_3 u_3(x_1, x_2) \tag{3.7}$$

$$\mathbf{v}(x_1, x_2) = \frac{\hat{e}_1}{h_2 h_3} \frac{\partial}{\partial x_2} \phi(x_1, x_2) - \frac{\hat{e}_2}{h_1 h_3} \frac{\partial}{\partial x_1} \phi(x_1, x_2) + \hat{e}_3 v_3(x_1, x_2) \tag{3.8}$$

Equations (3.7) and (3.8) transform equation (2.7) as

$$\frac{\hat{e}_1}{h_1 h_3}\left(v_3 \frac{\partial \psi}{\partial x_1} - u_3 \frac{\partial \phi}{\partial x_1}\right) + \frac{\hat{e}_2}{h_2 h_3}\left(v_3 \frac{\partial \psi}{\partial x_2} - u_3 \frac{\partial \phi}{\partial x_2}\right) + \frac{\hat{e}_3}{h_1 h_2 h_3^2}\left(\frac{\partial \psi}{\partial x_1}\frac{\partial \phi}{\partial x_2} - \frac{\partial \psi}{\partial x_2}\frac{\partial \phi}{\partial x_1}\right) = \mathbf{0} \tag{3.9}$$

Thus, from (3.9) we have respectively

$$\frac{1}{h_1 h_3}\left(v_3 \frac{\partial \psi}{\partial x_1} - u_3 \frac{\partial \phi}{\partial x_1}\right) = 0 \tag{3.10}$$

$$\frac{1}{h_2 h_3}\left(v_3 \frac{\partial \psi}{\partial x_2} - u_3 \frac{\partial \phi}{\partial x_2}\right) = 0 \tag{3.11}$$

$$\frac{1}{h_1 h_2 h_3^2}\left(\frac{\partial \psi}{\partial x_1}\frac{\partial \phi}{\partial x_2} - \frac{\partial \psi}{\partial x_2}\frac{\partial \phi}{\partial x_1}\right) = 0 \tag{3.12}$$

On writing (3.12) in Jacobian notation (A1), we have

$$\frac{1}{h_1 h_2 h_3^2}[\psi, \phi] = 0 \tag{3.13}$$

If we assume that the magnetic field is along a fixed direction (Suryanarayan, 1965), equation (2.6) will imply that

$$\mathbf{H} \cdot \text{grad } \mathbf{H} = 0 \tag{3.14}$$

and hence the momentum equation (2.2) for the fluid phase can be written as

$$\mathbf{u} \cdot \text{grad } \mathbf{u} = -\nabla\left(\frac{p}{\rho} + \frac{\mu}{2\rho}|\mathbf{H}|^2\right) + \frac{KN}{\rho}(\mathbf{v} - \mathbf{u}) \tag{3.15}$$

Using equation (2.5) in (3.15), we obtain

$$\text{curl } \mathbf{u} \times \mathbf{u} = -\nabla\left(\frac{p}{\rho} + \frac{1}{2}|\mathbf{u}|^2 + \frac{\mu}{2\rho}|\mathbf{H}|^2 + \frac{mN}{2\rho}|\mathbf{v}|^2\right) - \frac{mN}{\rho}\text{curl } \mathbf{v} \times \mathbf{v} \tag{3.16}$$

Substituting the values of **u** and **v** from (3.7) and (3.8) in (3.16), we can write respectively, in the following form:

$$\begin{aligned} \frac{\partial}{\partial x_1}\left(\frac{p}{\rho} + \frac{1}{2}|\mathbf{u}|^2 + \frac{\mu}{2\rho}|\mathbf{H}|^2 + \frac{mN}{2\rho}|\mathbf{v}|^2\right) &= \frac{1}{h_1 h_2 h_3}\left\{\frac{\partial \psi}{\partial x_1}\left[\frac{\partial}{\partial x_1}\left(\frac{h_2}{h_1 h_3}\frac{\partial \psi}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(\frac{h_1}{h_2 h_3}\frac{\partial \psi}{\partial x_2}\right)\right]\right. \\ &\quad \left. + \frac{mN}{\rho}\frac{\partial \phi}{\partial x_1}\left[\frac{\partial}{\partial x_1}\left(\frac{h_2}{h_1 h_3}\frac{\partial \phi}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(\frac{h_1}{h_2 h_3}\frac{\partial \phi}{\partial x_2}\right)\right]\right\} \\ &\quad + \frac{1}{2h_3^2}\left[\frac{\partial}{\partial x_1}(h_3 u_3)^2 + \frac{mN}{\rho}\frac{\partial}{\partial x_1}(h_3 v_3)^2\right] \end{aligned} \tag{3.17}$$

$$\begin{aligned} \frac{\partial}{\partial x_2}\left(\frac{p}{\rho} + \frac{1}{2}|\mathbf{u}|^2 + \frac{\mu}{2\rho}|\mathbf{H}|^2 + \frac{mN}{2\rho}|\mathbf{v}|^2\right) &= \frac{1}{h_1 h_2 h_3}\left\{\frac{\partial \psi}{\partial x_2}\left[\frac{\partial}{\partial x_1}\left(\frac{h_2}{h_1 h_3}\frac{\partial \psi}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(\frac{h_1}{h_2 h_3}\frac{\partial \psi}{\partial x_2}\right)\right]\right. \\ &\quad \left. + \frac{mN}{\rho}\frac{\partial \phi}{\partial x_2}\left[\frac{\partial}{\partial x_1}\left(\frac{h_2}{h_1 h_3}\frac{\partial \phi}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(\frac{h_1}{h_2 h_3}\frac{\partial \phi}{\partial x_2}\right)\right]\right\} \\ &\quad + \frac{1}{2h_3^2}\left[\frac{\partial}{\partial x_2}(h_3 u_3)^2 + \frac{mN}{\rho}\frac{\partial}{\partial x_2}(h_3 v_3)^2\right] \end{aligned} \tag{3.18}$$

and

$$\frac{\partial\psi}{\partial x_2}\frac{\partial}{\partial x_1}(h_3u_3)-\frac{\partial\psi}{\partial x_1}\frac{\partial}{\partial x_2}(h_3u_3)+\frac{\partial\phi}{\partial x_2}\frac{\partial}{\partial x_1}(h_3v_3)-\frac{\partial\phi}{\partial x_1}\frac{\partial}{\partial x_2}(h_3v_3)=0 \quad (3.19)$$

Writing (3.19) in Jacobian notation, we have

$$[h_3u_3, \psi]+[h_3v_3, \phi]=0 \quad (3.20)$$

Equations (3.7), (3.8), (3.13), (3.17), (3.18), and (3.20) are the basic equations of the present theory.

4. SOLUTIONS OF THE STEADY HYDROMAGNETIC TWO-PHASE EQUATIONS

Equations (2.1) and (2.4) have been solved and the result is given by equations (3.7) and (3.8), respectively.

Equation (3.13) yields

$$[\phi, \psi]=0 \quad (4.1)$$

The general solution of (4.1) is given by [cf. (A3)]

$$\phi=\phi(\psi) \quad (4.2)$$

Equation (3.19) with the help of (4.2) yields [cf. (A4)]

$$[h_3u_3+h_3\phi_\psi v_3, \psi]=0 \quad (4.3)$$

which has the general solution

$$h_3(u_3+\phi_\psi v_3)=\Omega(\psi) \quad (4.4)$$

where $\Omega(\psi)$ is an arbitrary function of ψ.

Differentiating partially (3.10) and (3.11) with respect to x_1 and x_2, respectively, and then subtracting, we get

$$\frac{\partial v_3}{\partial x_2}\frac{\partial\psi}{\partial x_1}-\frac{\partial v_3}{\partial x_1}\frac{\partial\psi}{\partial x_2}=\frac{\partial u_3}{\partial x_2}\frac{\partial\phi}{\partial x_1}-\frac{\partial u_3}{\partial x_1}\frac{\partial\phi}{\partial x_2} \quad (4.5)$$

This gives in Jacobian notation

$$[v_3-\phi_\psi u_3, \psi]=0 \quad (4.6)$$

The general solution of (4.6) is given by

$$v_3-\phi_\psi u_3=\xi(\psi) \quad (4.7)$$

where $\xi(\psi)$ is another arbitrary function of ψ. From (4.4) and (4.7), we obtain

$$v_3 = \frac{1}{1+\phi_\psi^2}\left[\frac{\Omega(\psi)}{h_3}\phi_\psi + \xi(\psi)\right] \tag{4.8}$$

$$u_3 = \frac{\Omega(\psi)}{h_3} - \phi_\psi \frac{1}{1+\phi_\psi^2}\left[\frac{\Omega(\psi)}{h_3}\phi_\psi + \xi(\psi)\right] \tag{4.9}$$

Now equation (3.16) can be written in the form

$$\nabla B = (\nabla \times \mathbf{v}) \times \mathbf{v} + \frac{\rho}{mN}\mathbf{u} \times (\nabla \times \mathbf{u}) \tag{4.10}$$

where

$$B = \frac{\rho}{mN}\left(\frac{p}{\rho} + \frac{1}{2}|\mathbf{u}|^2 + \frac{\mu}{2\rho}|\mathbf{H}|^2 + \frac{mN}{2\rho}|\mathbf{v}|^2\right)$$

Taking the dot product of $\mathbf{u}$ with (4.10), we get

$$\mathbf{u} \cdot \nabla B = \mathbf{u} \cdot (\nabla \times \mathbf{v}) \times \mathbf{v} \tag{4.11}$$

This can be manipulated into the form

$$[B, \psi] = \frac{1}{2h_3^2}\left\{[h_3^2 v_3^2, \psi] - \frac{u_3}{h_3}[h_3 v_3, \phi]\right\} \tag{4.12}$$

Using relation (3.20), we find that equation (4.12) becomes

$$[B, \psi] = \frac{1}{2h_3^2}\{[h_3^2 v_3^2, \psi] + [h_3^2 u_3^2, \psi]\} \tag{4.13}$$

Transforming to x_1-ψ coordinates [cf. (A7)], we get

$$\frac{\partial B(x_1, \psi)}{\partial x_1} = \frac{1}{2h_3^2}\left\{\frac{\partial}{\partial x_1}[h_3(x_1, \psi) v_3(x_1, \psi)]^2 + \frac{\partial}{\partial x_1}[h_3(x_1, \psi) u_3(x_1, \psi)]^2\right\}$$

where in taking the derivatives we take ψ fixed. Manipulating the right-hand side of this equation with the help of (4.4), we can write it as a partial derivative with respect to x_1 for ψ held constant:

$$\frac{\partial}{\partial x_1} B(x_1, \psi) = \frac{\xi(\psi)}{h_3}\frac{\partial}{\partial x_1}(h_3 v_3) \tag{4.14}$$

By the use of equation (4.8) we obtain

$$\frac{\partial B(x_1, \psi)}{\partial x_1} = \frac{\xi^2(\psi)}{1+\phi_\psi^2}\frac{\partial}{\partial x_1}(\ln h_3) \tag{4.15}$$

The general solution of equation (4.15) is

$$B = \frac{\xi^2(\psi)}{1+\phi_\psi^2} \ln h_3 + F(\psi)$$

where $F(\psi)$ is a definite function of ψ.

APPENDIX

Here we give for the convenience of the reader some elementary, but very useful differential relations which we have used in the solution of the two-phase hydromagnetic equations. By definition, the Jacobian of two functions $f(x_1, x_2)$ and $g(x_1, x_2)$ is

$$[f, g] = \frac{\partial f}{\partial x_1}\frac{\partial g}{\partial x_2} - \frac{\partial f}{\partial x_2}\frac{\partial g}{\partial x_1} \tag{A1}$$

The following relations are self-evident:

$$[f, g] = -[g, f] \tag{A2}$$

$$[f, g] = 0 \Rightarrow f = f(g) \tag{A3}$$

$$[f, g+G] = [f, g] + [f, G] \tag{A4}$$

$$[f, gG] = [f, g]G + g[f, G] \tag{A5}$$

Let $S(x_1, x_2)$ and $A(x_1, x_2)$ represent two arbitrary functions of the variables x_1 and x_2. The following are also evident:

$$\frac{\partial S(x_1, x_2)}{\partial x_1} = \frac{\partial A(x_1, x_2)}{\partial x_1}\frac{\partial S(x_2, A)}{\partial A} \tag{A6}$$

$$\frac{\partial S(x_1, x_2)}{\partial x_2} = \frac{\partial A(x_1, x_2)}{\partial x_2}\frac{\partial S(x_1, A)}{\partial A} \tag{A7}$$

where we have denoted the argument of each function to be regarded as independent variables.

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